1. Evaluate the following integrals:

(a) $\int_1^2 (\frac{1}{t} - i)^2 \, dt$

(b) $\int_0^{\pi/6} e^{i2t} \, dt$.

Solution:

(a)

$$\int_1^2 (\frac{1}{t} - i)^2 \, dt = \int_1^2 (\frac{1}{t^2} + i^2 - 2i\frac{1}{t}) \, dt$$

$$= \int_1^2 (\frac{1}{t^2} - 1) \, dt - 2i \int_1^2 \frac{1}{t} \, dt$$

$$= -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4.$$

(b) $$\int_0^{\pi/6} e^{i2t} \, dt = \left. \frac{e^{i2t}}{2i} \right|_0^{\frac{\pi}{6}} = \frac{1}{2i} (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1)$$

$$= \frac{\sqrt{3}}{4} + \frac{i}{4}.$$

2. $f(z) = \dfrac{z+2}{z}$ and C is

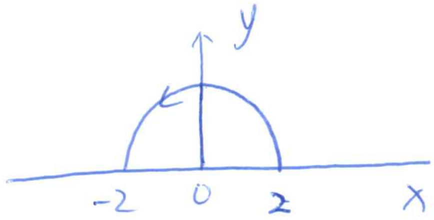
(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Solution

(a) Let C be the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$, shown below



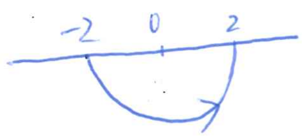
Then

$$\int_C \frac{z+2}{z}\,dz = \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^{\pi} \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta}\,d\theta$$

$$= 2i \int_0^{\pi} (e^{i\theta} + 1)\,d\theta = 2i \left[\frac{e^{i\theta}}{i} + \theta\right]_0^{\pi}$$

$$= 2i\,(i + \pi + i) = -4 + 2\pi i$$

(b) Let C be the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$, shown below



$$\int_C \frac{z+2}{z}\,dz = \int_{\pi}^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta}\,d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_{\pi}^{2\pi} = 4 + 2\pi i$$

Solution:

(a) $z^{-3/4} = \exp[-\frac{3}{4}(\ln 1 + i\theta)]$ $\quad |z| > 0, -\pi < \mathrm{Arg}\, z < \pi$

$= e^{-\frac{3}{4}i\theta}$

Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} e^{-\frac{3}{4}i\theta}\, i e^{i\theta}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{\frac{1}{4}i\theta}\, d\theta = i \left.\frac{e^{\frac{1}{4}i\theta}}{\frac{1}{4}i}\right|_{-\pi}^{\pi}$$

$$= 4[e^{\frac{i}{4}\pi} - e^{-\frac{i}{4}\pi}] = 4\sqrt{2}i$$

(b) $z^{-3/4} = \exp[-\frac{3}{4}(\ln 1 + i\theta)]$ $\quad |z| > 0, \quad 0 < \arg\theta < 2\pi$

$= e^{-\frac{3}{4}i\theta}$

Let C be the unit circle $z = e^{i\theta}$, $\quad 0 \le \theta \le 2\pi$.

$$\int_C f(z)\,dz = \int_0^{2\pi} e^{-\frac{3}{4}i\theta}\, i e^{i\theta}\, d\theta = 4(e^{\frac{\pi}{2}i} - 1)$$

$= -4 + 4i$

(c). Finally, let C denote the entire circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$. In this case, $\int_C \frac{z+2}{z} dz = 4\pi i$, the value here being the sum of the values of the integers in parts (a) and (b).

3. Let C denote the positive oriented unit circle $|z|=1$ about the origin.

(a) Show that if $f(z)$ is the principal branch

$$z^{-3/4} = \exp[-\tfrac{3}{4} \operatorname{Log} z] \quad (|z| > 0, \ -\pi < \operatorname{Arg} z < \pi)$$

of $z^{-3/4}$, then $\int_C f(z) dz = 4\sqrt{2} i$

(b) Show that if $g(z)$ is the branch

$$z^{-3/4} = \exp[-\tfrac{3}{4} \log] \quad (|z| > 0, \ 0 < \arg z < 2\pi)$$

of the same power function as part (a), then

$$\int_C g(z) dz = -4 + 4i$$

This exercise demonstrate how the value of an integral of a power function depends in general on the branch that it used.

4. Show that

$$\int_{C_0} (z-z_0)^{n-1}\, dz = 0 \qquad (n = \pm 1, \pm 2, \cdots)$$

when C_0 is any closed contour does not pass through the point z_0.

Solution

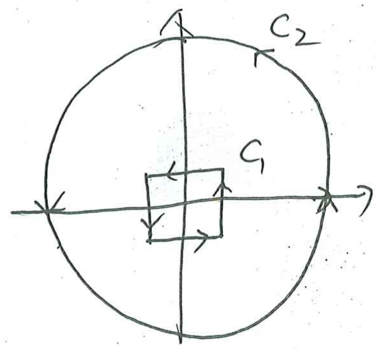
Note that $(z-z_0)^{n-1}$ $(n = \pm 1, \pm 2, \cdots)$ always has an antiderivative in any domain that does not contain the point $z = z_0$. So by theorem in Section 48.

$$\int_{C_0} (z-z_0)^{n-1}\, dz = 0 \qquad (n = \pm 1, \pm 2, \cdots).$$

for any closed contour C_0 does not pass through z_0.

5. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Point out why

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$



when (a) $f(z) = \frac{1}{3z^2+1}$; (b) $f(z) = \frac{z+2}{\sin(z/2)}$; (c) $f(z) = \frac{z}{1-e^z}$

Solution:

In each of case below, the singularities of the integrand lie outside C_1 or inside C_2; and so the integrand is analytic on the contours and between them. Consequently $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

(a) when $f(z) = \frac{1}{3z^2+1}$, the singularites are the points $z = \pm\frac{1}{\sqrt{3}} i$

(b) when $f(z) = \frac{z+2}{\sin(z/2)}$, the singularities are at $z = 2n\pi$ ($n = 0, \pm 1, \cdots$)

(c) when $f(z) = \frac{z}{1-e^z}$, the singularities are at $z = 2n\pi i$ ($n = 0, \pm 1, \pm 2, \cdots$)

6. Let C_R denote the upper half of the circle $|z|=R$ $(R>2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \right| \leq \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}.$$

Solution:

Note that $|z|=R$, $R>2$ then

$$|2z^2-1| \leq 2|z|^2+1 = 2R^2+1$$

and

$$|z^4+5z^2+4| = |(z^2+4)||(z^2+1)| \geq \left||z|^2-1\right| \left||z|^2-4\right|$$

$$=(R^2-1)(R^2-4)$$

Thus

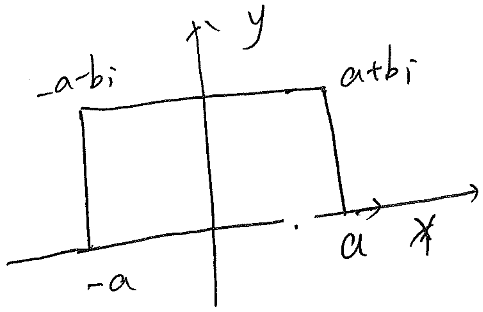
$$\left|\frac{2z^2-1}{z^4+5z^2+4}\right| = \frac{|2z^2-1|}{|z^4+5z^2+4|} \leq \frac{2R^2+1}{(R^2-1)(R^2-4)}$$

Since the length of C_R is πR. then

$$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \right| \leq \pi R \cdot \frac{2R^2+1}{(R^2-1)(R^2-4)}$$

Hints about HW 3. P_{159} 4.6.

Q4. In order to derive the integration formula in question, we integrate the function e^{-z^2} around the closed rectangular path shown below



Since the lower horizontal leg is represented by $z=x$, $-a\le x\le a$, the integration of e^{-z^2} along that leg is $\int_{-a}^{a} e^{-x^2}dx$

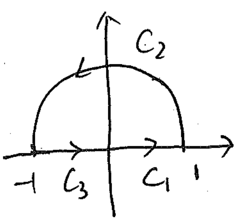
The opposite direction of upper horizontal leg has parametric representation $z=x+ib$, $-a\le x\le a$, the integral of e^{-z^2} along upper leg is

$$-\int_{-a}^{a} e^{-(x+bi)^2}dx.$$

The right hand vertical leg is represented by $z=a+iy$, $0\le y\le b$.

$$\int_0^b e^{-(a+iy)^2}\, i\,dy$$

Q6. Let C denote the entire boundary of the semicircular region below.



On C_1 : $z=re^{i0}$, $0\le r\le 1$

on C_2 : $z=1\cdot e^{i\theta}$, $0\le\theta\le\pi$

On C_3 : $z=re^{i\pi}$, $0\le r\le 1$

$C=C_1+C_2+C_3$